Large School Team Test – Round 11003

Unless otherwise indicated, leave each answer in an exact and simplified form.

1. Find the value of b for which the lines $6x + by = 4$ and $4y + 2x = b$ do not intersect.

2. Find the area of an equilateral triangle inscribed in a circle of radius 1.

3. I have two boxes of marbles. The first has 2009 black marbles and 39 white marbles. The second has 1319 black marbles and 729 white marbles. I randomly choose three marbles out of each box and put them into a third box. Then I randomly draw a marble out of the third box. What is the probability that this marble is white?

4. If $\cos(2x) = 2\cos(x)$, find $\sin^2(x)$.

5. There are 2009 people in a room, each of whom either always lies or always tells the truth. If each one of them accuses one other person in the room of being a liar, what is the maximum number of liars that could be in the room?

6. Square ABCD has side length 2. A circle is tangent to side CD and passes through A and B. Find the radius of the circle.

7. How many noncongruent right triangles are there with integer sides, and one leg equal to 12?

8. Triangle ABC has AB=6, BC=18, and $\angle ABC = 30°$. Find the area of the triangle.

9. 12 teams play a tournament in which each team plays every other team once, and there are no ties. How many games must my team win to be certain of finishing no worse than a tie for fifth?

10. Find all real values of x for which the equation $y^3 - x^2y - 3y^2 + x^2 + 3y - 1 = 0$ has exactly one real solution for y.

Large School Team Test – Round 11003 (Solutions)

Answers must be exact and simplified. (But slightly different forms of answers are acceptable.)

1. We want the two lines to be parallel — that is, have the same slope. The slope of the first line is -6/b, and the slope of the second is -1/2, so b is 12.

12

2. For any regular n-gon inscribed in a circle of radius r, we can partition the figure into triangles by drawing segments from the center of the circle to the vertices of the figure, each of which are radii of the circle. The area of each of these triangles is given by the formula $\frac{1}{2}ab\sin C$, where C is a central angle of the circle, and so the total area of the n-gon is $\frac{nr^2}{2}\sin\left(\frac{2\pi}{n}\right)$. In this particular case, we have r = 1 and n = 3, so the area is $\frac{3}{2}\sin\left(\frac{2\pi}{3}\right)=\frac{3\sqrt{3}}{4}$.

$\mathbf{\frac{3\sqrt{3}}{4}}$

3. Because both boxes initially contain the same number of marbles, each marble has an equal probability of being chosen as the final one. There are 768 total white marbles, and 4096 total marbles, so the probability is 768/4096 = 3/16.

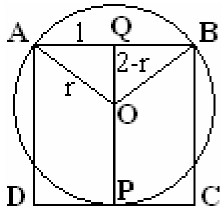
$\mathbf{\frac{3}{16}}$

4. Let C = cos(x). Then the equation given becomes $2C^2 - 1 = 2C$, which has the solution $C=\frac{2\pm\sqrt{12}}{4}=\frac{1-\sqrt{3}}{2}$ (the other solution is larger than 1). Then $\sin^2(x)=1-C^2=1-\frac{4-2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$.

$\mathbf{\frac{\sqrt{3}}{2}}$

5. Suppose all 2009 people are liars. Then they must each accuse someone else—correctly—of being a liar. This makes them all truth-tellers, so it is impossible. But if 2008 of them are liars, they can all accuse the single truth-teller of being a liar, so this works.

2008

6. 

Let O be the center of the circle, and P be the point of tangency with side CD. P is the midpoint of CD, and let Q be the midpoint of AB. AQO forms a right triangle. AQ = 1 and AO is a radius. PQ = 2 and PO is a radius, so OQ = 2 – r. So $1 + (2 - r)^2 = r^2$. Expanding this gives $5 – 4r + r^2 = r^2$, so r = 5/4.

$$\mathbf{\frac{5}{4}}$$

7. Let the other leg be b, and the hypotenuse be a. Then $144 = a^2 - b^2$. Factoring this gives $2^4 \cdot 3^2 = (a - b)(a + b)$. a - b and a + b are both even or both odd, so they must both be even in this case. So, dividing by 4, we have $2^2 \cdot 3^2 = \frac{a-b}{2} \cdot \frac{a+b}{2}$, where $\frac{a-b}{2}$ and $\frac{a+b}{2}$ are both integers. We can choose $\frac{a-b}{2}$ to be any of the factors of 36, as long as $\frac{a+b}{2}$ is strictly larger. The possible choices are 1, 2, 3, and 4. (We could choose 6, but this would make b = 0.) Each of these yields a unique triangle when we solve for a and b, so there are four possible triangles. Their side lengths are 5-12-13, 9-12-15, 12-16-20, and 12-35-37.

4

8. The area of any triangle is $\frac{1}{2}ab\sin C$, where a and b are two sides, and C is the angle between them.

Applying this to the triangle we have, we get $A = \frac{1}{2} * 6 * 18 * \sin 30 = 27$.

27

9. In order to finish no worse than a tie for fifth, my team must ensure that it is impossible for 5 teams to get more wins than us. Given any set of 5 teams, they play 10 games among themselves, so at least one of them must have 2 losses. Thus 9 wins (out of 11 games) will certainly be enough. In fact, if we only win 8 games, we still only have 3 losses, so among any set of 5 other teams, we must win against at least 2 of them. That gives those 5 teams 12 losses between them, so one of them must have at least 3 losses (and at most 8 wins). So 8 wins are enough.

But if we only have 7 wins, it is easy to construct a scenario in which we finish sixth or worse. A set of 5 other teams can have 2 wins and 2 losses against other teams from that set, and each of them can win their other 7 games, except for possibly losing to my team. This gives each of those 5 teams at least 8 wins.

8

10. **<u>Method I:</u>** Rewrite the equation as $(y^3 - 3y^2 + 3y - 1) + (-x^2y + x^2) = (y - 1)^3 - x^2(y - 1) = (y - 1)((y - 1)^2 - x^2) = 0$. This always has the real root y = 1. Additionally, it will also have the roots $y = 1 \pm x$. In order to give exactly one real solution, x must be zero.

<u>Method II:</u> If we let x = 0, we have $y^3 - 3y^2 + 3y – 1 = (y – 1)^3 = 0$. If we let y = 1 in the original equation, we find that $1 - x^2 - 3 + x^2 + 3 – 1$ identically equals zero, so 1 is always a root. So, dividing the equation by y – 1 (synthetic division is easiest), we have the quadratic $y^2 – 2y + 1 – x^2 = 0$, and its discriminant is $4 – 4 + x^2 = x^2$, which is always positive (except when x = 0, which we have already explored), giving more than one distinct real root. So zero is the only possible value for x.

0