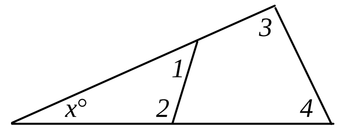
Target Test – Round 11003

Unless otherwise indicated, leave each answer in an exact and simplified form.

mathleague.org – Target Round 11003.1



1. Two overlapping triangles share the angle measuring $x°$ as shown. If $m\angle 3 = 2 \cdot m\angle 1$, $m\angle 4$ is 40° less than $m\angle 3$, and $m\angle 2$ is 49° more than $m\angle 4$, then find x.

2. A *Farey sequence* is the set of all rational numbers between 0 and 1, inclusive, written in order as reduced fractions, with a maximum denominator of some integer value, n. For example, if $n = 4$, then its Farey sequence is $F_4 = \{0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1\}$. This can be written in general as $F_n = \{f_1, f_2, f_3, f_4, \ldots, f_k\}$. Find the sum of the series $-f_1 + f_2 - f_3 + f_4 - f_5 + \ldots + f_{k-1} - f_k$ for F_6.

mathleague.org – Target Round 11003.2

3. The term *googol* was coined as a fanciful name for the number 10^{100}. A *googolplex* is 10 to the googol$^{\text{th}}$ power, $10^{\left(10^{100}\right)}$, a number so large that it cannot be written out in standard notation, in base ten. However, a "binary" googolplex can. Convert $10^{\left(10^{100}\right)}{}_2$ (base two) to base ten.

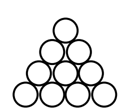
4. The latest electronic toy for toddlers, the iSpud, was released on 1/31/08. Due to its popularity, dealers are able to raise the price by 25% each year in time for Christmas rush. Because of decreasing costs, they can then comfortably lower the price by 50% right after Jan. 1 each year. Based on this, what fraction of the original price will the iSpud be selling for on 12/31/10?

mathleague.org – Target Round 11003.3

5. If $\sqrt{6 + x} - 2x\sqrt{3} = 0$, find all real solutions for x.

6. The triangular numbers, {1, 3, 6, 10, 15, …}, are used to generate an irrational decimal k as follows: the first decimal digit is 0, the next 3 digits are '123', the next 6 are 0, the next 10 are repeating '123's, etc. So, k = .01230 00000 12312 31231 00000 00000 00000 1 … . Find the sum of the 101st to 200th decimal digits of k.

mathleague.org – Target Round 11003.4



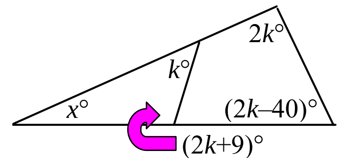
7. A triangular array of 10 congruent spheres are arranged to be tangent as shown, such that their centers and points of contact are all coplanar. If each sphere has a radius of 1, then what is the volume of the smallest triangular prism which will contain them?

8. When n standard six-sided dice are rolled, the probability of rolling a sum of 2009 is greater than zero, and is the same as the probability of rolling a sum of S. What is the smallest possible value of S?

Target Test – Round 11003
(Solutions)

Answers must be exact and simplified. (But slightly different forms of answers are acceptable.)

1. Using the givens, relabel the figure as shown. Then create the equations $x + k + 2k + 9 = 180$ and $x + 2k + 2k - 40 = 180$. Subtracting the first equation from the second gives $k - 49 = 0$ $\Rightarrow$ $k = 49$, and then substitute to find x: $x + 3 \cdot 49 + 9 = 180 \Rightarrow$ $x + 156 = 180 \Rightarrow x = \boxed{24}$.



24

2. **<u>Method I:</u>** The sequence $F_6 = \{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\}$. The series is

$$-\frac{0}{1} + \frac{1}{6} - \frac{1}{5} + \frac{1}{4} - \frac{1}{3} + \frac{2}{5} - \frac{1}{2} + \frac{3}{5} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \frac{1}{1} = \left(\frac{1}{6}+\frac{1}{4}+\frac{2}{5}+\frac{3}{5}+\frac{3}{4}+\frac{5}{6}\right) - \left(\frac{0}{1}+\frac{1}{5}+\frac{1}{3}+\frac{1}{2}+\frac{2}{3}+\frac{4}{5}+\frac{1}{1}\right).$$

After noting that the three like-denominator pairs in each pair of parentheses add up to one, <u>except</u> for the $\frac{1}{2}$, have $3 - 3\frac{1}{2} = \boxed{-.5}$.

<u>Method II:</u> Examining simpler sequences: $F_2 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$ yields the series $-\frac{0}{1} + \frac{1}{2} - \frac{1}{1} = \frac{1}{2} - 1 = -\frac{1}{2}$; $F_3 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$ yields $-\frac{0}{1} + \frac{1}{3} - \frac{1}{2} + \frac{2}{3} - \frac{1}{1}$ = [by pairing like signs] $(\frac{1}{3} + \frac{2}{3}) - (\frac{1}{2} + \frac{1}{1}) = 1 - 1\frac{1}{2} = -\frac{1}{2}$; and $F_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$ yields $-\frac{0}{1} + \frac{1}{4} - \frac{1}{3} + \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{1}{1} = (\frac{1}{4} + \frac{3}{4} + \frac{1}{2}) - (\frac{1}{3} + \frac{2}{3} + \frac{1}{1}) = 1\frac{1}{2} - 2 = -\frac{1}{2}$ once again. Based on this pattern, $\boxed{-.5}$ is a good guess for F_6.

<u>Method III:</u> [*the long way*] The series generates the pairs $-\frac{0}{1} + \left(\frac{1}{6} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{3}\right) + \left(\frac{2}{5} - \frac{1}{2}\right)$ $+\left(\frac{3}{5} - \frac{2}{3}\right) + \left(\frac{3}{4} - \frac{4}{5}\right) + \left(\frac{5}{6} - \frac{1}{1}\right) = 0 - \frac{1}{30} - \frac{1}{12} - \frac{1}{10} - \frac{1}{15} - \frac{1}{20} - \frac{1}{6}$ [getting all unit fractions is <u>not</u> a coincidence] $= 0 - \frac{2}{60} - \frac{5}{60} - \frac{6}{60} - \frac{4}{60} - \frac{3}{60} - \frac{10}{60} = 0 - \frac{30}{60} = \boxed{-.5}$.

$-\frac{1}{2}$

3. $10^{\left(10^{100}\right)}{}_{2} = 2^{\left(2^{4}\right)}{}_{10} = 2^{16} = \boxed{65\ 536}$.

65 536

4. Near the end of each year, the price goes up 25%, or is multiplied by $125\% = \frac{5}{4}$. On Jan. 1, it is multiplied by $\frac{1}{2}$. If it begins as x, then it changes to $x \cdot \frac{5}{4} \cdot \frac{1}{2} \cdot \frac{5}{4} \cdot \frac{1}{2} \cdot \frac{5}{4} = \frac{125}{256}x$. So the fraction is $\boxed{\frac{125}{256}}$.

$\frac{125}{256}$

5. $\sqrt{6+x} - 2x\sqrt{3} = 0 \Rightarrow \sqrt{6+x} = 2x\sqrt{3} \Rightarrow 6 + x = 4x^2 \cdot 3 \Rightarrow$
$12x^2 - x - 6 = 0 \Rightarrow (4x-3)(3x+2) = 0 \Rightarrow x = \frac{3}{4}$ or $-\frac{2}{3}$. So, $x = \boxed{\frac{3}{4}}$.

$\frac{3}{4}$

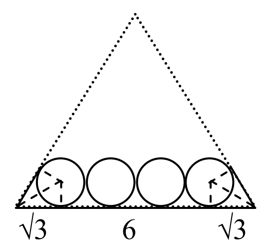
6. The table below shows the first 10 triangular numbers and their accumulating sums:

n	1	2	3	4	5	6	7	8	9	10
t_n	1	3	6	10	15	21	28	36	45	55
S_n	1	4	10	20	35	56	84	120	165	220

Since the odd-n triangulars have zeros, the 85th to 120th digits for the 8th triangular have '123's repeating. But we wish to start at the 101st digit, which is 16 or 5·3+1 after d_{85}, so $d_{101} = 2$. Hence, for d_{101} to d_{120}, we have 6 '231's and a '23', so the sum from the 8th triangular is $6 \cdot 6 + 5 = 41$. After the 9th triangular's 0's through d_{165}, we find d_{166} to d_{200} have 11 '123's and a '12', adding $11 \cdot 6 + 3 = 69$. So, the sum is $S = 41 + 69 = \boxed{110}$.

110

7. The edge length of each triangular face is found by noting that (see figure) the distance across the 4 centers is 6, and each angle subtends 30:60:90 triangles with a longer leg of $\sqrt{3}$; so the edge of a triangular face is $6 + 2\sqrt{3}$. The height of the prism is 2. The area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, so the volume of the prism is $\frac{2\left(6+2\sqrt{3}\right)^2\sqrt{3}}{4} =$
$\frac{\left(36+24\sqrt{3}+12\right)\sqrt{3}}{2} = \boxed{36+24\sqrt{3}}$.



$36 + 24\sqrt{3}$

8. **<u>Method I:</u>** For n dice, the sum S will be in the range $n \le S \le 6n$. Due to symmetry in the number of ways to roll all of the possible sums [e.g., for 2 dice, the number of ways of rolling 2 and 12 are the same, as are 3 and 11, 4 and 10, etc.], the probability of rolling $n + k$ is the same as that of rolling $6n - k$. To make S as small as possible, must have n as small as possible and make $6n - k = 2009 \Rightarrow 6n = 2009 + k$ while $S = n + k$. Since $6(335) = 2010$, $n = 335$ and $k = 1$. So, $S = 335 + 1 = \boxed{336}$.

<u>Method II:</u> On a standard die, 6 and 1, 5 and 2, and 4 and 3 are on opposite sides. The fewest dice will result in the smallest S, so as many sixes as possible are needed to generate a total roll of 2009, thus providing the maximum number of ones appearing on the bottom (which would have the same probability as being on the top). Dividing 2009 by 6 yields 334 with remainder 5, so there will be 334 ones and a two showing, or $334(1) + 1(2) = \boxed{336}$.

336