mathleague.org Sprint Test – Round 11003

1. What is the smallest prime number p such that p - 1 has exactly two distinct prime factors?

   A) 2 B) 3 C) 5
   D) 7 E) None of these

2. Let f(n) be the number of positive integers in the set {1, 2, ..., n} that are relatively prime to n. What is the largest positive integer less than 100 such f(n) = n - 1?

   A) 89 B) 91 C) 95
   D) 99 E) None of these

3. If a fair coin is flipped 3 times, what is the probability of throwing heads twice in a row?

   A) 1/2 B) 3/8 C) 5/8
   D) 1/4 E) None of these

4. Fifteen people attend driving school. To attend, a student must be at least 16 years of age. The average age of ten of the students is 18, and the average age of all the students is 19. What is the maximum possible age of the oldest student?

   A) 26 B) 31 C) 36
   D) 41 E) None of these

5. 88 eighths of 8 is 8 eighty-eighths of

   A) 888 B) 9986 C) 12100
   D) $88 \cdot 888$ E) None of these

6. The volume of a cube is 8 times its surface area. What is the side length of the cube?

   A) 8 B) 16 C) 36
   D) 48 E) None of these

7. How many points (x,y) with integer coordinates lie on the circle centered at (0,0) with radius 5 ?

   A) 4 B) 8 C) 12
   D) 16 E) None of these

8. If $a > b$, $a > d$, $b > e$, and $c > d$, which of the following chains of inequalities is impossible?

   A) $a > d > e$
   B) $a > b > e$
   C) $a > c > e$
   D) $e > d$
   E) All are possible

9. Let ABC be a right angled triangle with right angle at B. Let D be the foot of the perpendicular from B onto AC. If AB = 9, and BC = 12, what is BD?

   A) 36/5 B) 20 C) 45/12
   D) 12 E) None of these

10. A function f(x) satisfies $f(2x) = 2f(x) – 1$. If $f(4) = 15$, what is f(1)?

    A) 3 B) 10 C) 41
    D) 57 E) None of these

11. What is the largest integer n such that 35^n divides 100! ?

A) 1 B) 2 C) 16
D) 24 E) None of these

12. If f(x) is a polynomial of degree 9 and has exactly 3 real roots, what is the maximum number of irreducible quadratic factors of f(x) with real coefficients?

A) 1 B) 2 C) 3
D) 4 E) None of these

13. How many subsets of {1, 2, 3, 4} are not subsets of {2, 3}?

A) 4 B) 8 C) 10
D) 12 E) None of these

14. How many nonempty subsets of the set {1, 2, 3, 4, 5, 6} have an even number of elements?

A) 3 B) 16 C) 31
D) 32 E) None of these

15. Let Θ be the acute angle, in degrees, between the lines y = 2x and y = 3x. What range is Θ in?

A) [0,10) B) [10,20) C) [20,30)
D) [30,40) E) None of these

16. How many 7-digit base 2 numbers have an even number of 1s in them?

A) 24 B) 32 C) 36
D) 40 E) None of these

17. What is the probability that two randomly selected edges of a regular 11-gon do not share a common vertex?

A) 1/2 B) 2/3 C) 3/4
D) 4/5 E) None of these

18. The vertices of an octagonal prism are colored in such a way that any two adjacent vertices have different colors. What is the minimum number of colors needed to accomplish this?

A) 2 B) 3 C) 4
D) 5 E) None of these

19. How many distinct ways are there to paint the faces of a cube with at most 2 colors if rotations are considered the same?

A) 5 B) 10 C) 12
D) 16 E) None of these

20. How many four digit positive integers are there with the property that the sum of the squares of their digits is 20?

A) 8 B) 10 C) 12
D) 14 E) None of these

21. How many ways are there to place checkers on a 4x4 checkerboard, one checker per cell, so that each row and each column has exactly 3 checkers in it?

A) 12 B) 18 C) 24
D) 30 E) None of these

22. An isosceles triangle has integer side lengths and perimeter 16. What is the largest possible value of its area?

A) 16 B) 12 C) $8\sqrt{2}$
D) $4\sqrt{3}$ E) None of these

23. Let N be the smallest positive integer such that both N and N + 1 have more than four distinct positive factors. What is the sum of the digits of N?

A) 7 B) 8 C) 9
D) 10 E) None of these

24. The second term of an infinite geometric series is 19. What is the smallest possible positive value of its sum?

A) 38 B) 57 C) 76
D) 95 E) None of these

25. For which of the following functions f(x) does the equation f(x) = x have more than one solution?

A) cos(x) B) sin(x) C) tan(x)
D) All of these E) None of these

26. Evaluate 1/2 – 2/4 + 3/8 – 4/16 + 5/32 - ...

A) 2/9 B) 3/8 C) 3/13
D) 5/12 E) None of these

27. An "increasing number", such as 23589, is a positive integer, each digit of which is larger than the digit to its left. When all five-digit increasing numbers are arranged from smallest to largest, the 105^{th} number in the list does not contain the digit

A) 5 B) 6 C) 7
D) 8 E) None of these

28. 8 identical spheres each of diameter 20 fit tightly into a cube of side length 40 so that each sphere just touches three of the faces of the cube. What is the radius of the largest sphere that will fit in the central space, just touching all eight other spheres, to the nearest tenth?

A) 7 B) 7.3 C) 7.6
D) 7.9 E) None of these

29. Triangles ABC and ABD are isosceles with AB = AC = BD, and BD intersects AC at E. If BD is perpendicular to AC, what is the sum, in degrees, of the angles ADB and ACB?

A) 115 B) 120 C) 130
D) 135 E) Not uniquely determined

30. Let N be the number of positive integers c such that the equation 14x + 64y = c has precisely one pair of positive integers (x,y) as a solution. What is N?

A) 112 B) 224 C) 448
D) 896 E) None of these

Sprint Test – Round 11003 (Solutions)

1. D	6. D	11. C	16. B	21. C	26. A
2. E (97)	7. C	12. C	17. D	22. B	27. A
3. B	8. E	13. D	18. A	23. B	28. B
4. D	9. A	14. C	19. B	24. C	29. D
5. E (968)	10. E (9/2)	15. A	20. C	25. C	30. B

9. A quick solution is to consider the area of the triangle in two different ways. Since the hypotenuse is 15, if d is the length of the perpendicular, calculating the area gives 9(12)/2 = 15d/2.

12. If f has 3 real roots then it can have 6 complex roots, occurring in 3 conjugate pairs

16. The first digit must be 1, so we need an odd number of 1s in the other 6 digits. C(6,1) + C(6,3) + C(6,5) = 32.

19. Just consider one color. We can color anywhere from 0 to 6 faces with that color. There is also a symmetry, e.g. 2 faces with one color means 4 faces with the other color. There is one possibility with 0 faces, and one possibility with 1 face. For 2 faces, they can either be adjacent or opposite. For 3, they can either all 3 be adjacent or 2 of them can be opposite and both adjacent to a third. 1 + 1 + 2 + 2 + 2 + 1 + 1 = 10.

21. This is the same as the number of ways of leaving exactly one blank square in each row and each column, or 4!

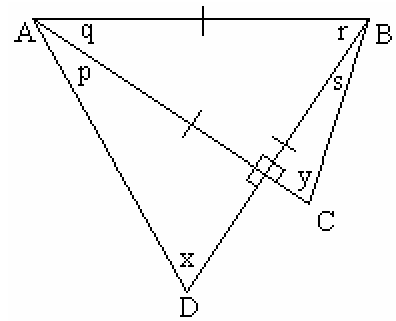
23. If we have exactly one prime factor, a, then it must be repeated 4 times (a^4) to have more than 4 factors. If we have exactly two prime factors, by themselves they are not enough, but if one is repeated (a^2b), then we have enough. If we have 3 (or more) prime factors (abc), it is good enough. So let's pick a maximum number, say 50, and increase it later if we need to. We want to list fourth powers of primes, squares of primes multiplied by another prime, products of 3 primes, and all their multiples. 2, 3, 5, 7, and 11 are good enough for now. We have $2^4 = 16$, (and 32, 48), 81 is too big; we have $2^2\cdot 3 = 12$ (24, 36, 48), $2^2\cdot 5 = 20$ (40), $2^2\cdot 7 = 28$, $2^2\cdot 11 = 44$; we have $3^2\cdot 2 = 18$ (36), $3^2\cdot 5 = 45$; we have $5^2\cdot 2 = 50$; then we have $2\cdot 3\cdot 5 = 30$ and $2\cdot 3\cdot 7 = 42$. If we list these in increasing order we find 44 and 45.

24. 19 = ar and hence the sum of the series is $\frac{a}{1-r} = \frac{ar}{r(1-r)} = \frac{19}{r(1-r)}$. The maximum value of r(1 - r) is 1/4, achieved when r = 1/2.

27. If we fix the first digit as 1, there are C(8,4) = 70 increasing numbers. If we fix the first digit as 2, there are C(7,4) = 35 increasing numbers, so in ascending order, the 105^{th} is 26789.

28. The distance between the centers of 2 diagonally opposite spheres is $20\sqrt{3}$ (from the distance formula in 3 dimensions), so the diameter of the center sphere is $20\sqrt{3}-20$, and the radius is then $10\left(\sqrt{3}-1\right)$.

29. Let x = angle ADB and y = angle ACB, as shown in the diagram, and p, q, r, and s as shown. We are looking for x + y. AC and BD partition the figure into three right triangles, and so p + x = q + r = s + y = 90. Adding all three of these together we have p + q + r + s + x + y = 270. Since AB = AC, we have r + s = y, and since AB = BD, we have p + q = x. Adding these together we have p + q + r + s = x + y. Substituting, we have 2(x + y) = 270, and so x + y = 135.



30. If we graph the family of lines 14x + 64y = c, each line has slope -14/64 = -7/32. We want to know how many of these lines pass through exactly one lattice point in the first quadrant, excluding the axes. This is equivalent to counting the number of such lattice points which yield such a line. Note that if x and y are positive integers, c will also be a positive integer. If a line with slope -7/32 passes through a lattice point (a, b) then it will also pass through (a – 32, b + 7) and (a + 32, b – 7). More specifically, these will be the *closest* points to (a, b) which the line passes through, because gcd(7, 32) = 1. So if both of these 2 points are not in the first quadrant, and (a, b) itself *is*, then we have a solution. Since the line has negative slope and must pass through the first quadrant, it also passes through the second and fourth quadrants. We want (a – 32, b + 7) to be in the second quadrant and (a + 32, b – 7) to be in the fourth quadrant, or on the axes, so $a - 32 \le 0$ and $b - 7 \le 0$. Also $a > 0$ and $b > 0$. So we have $1 \le a \le 32$ and $1 \le b \le 7$. This yields $7 \cdot 32 = 224$ solutions.

mathleague.org

Sprint Test Round xxxxx
(Solution Template)

Instructions for Use

Option I: You may use this page as-is and compare the answers side by side with students' papers. Count the number of correct answers and incorrect answers and write both numbers in the appropriate space on each student's answer sheet. Multiply the number of correct answers by 4 and subtract the number of incorrect answers to get the total score. Any item with more than one answer bubbled is counted as incorrect.

Option II: (Please note that although this option is faster it will only work if the test you are grading does not have any items with more than one answer bubbled.) The circles below are designed to be the same size as the holes created by a standard hole punch, so if you punch out all the correct answers on this page you should be able to lay this template over each student's answer sheet to count the number of correct answers quickly. Then you can remove the template and count how many total answers were given; the total minus the number of correct answers is of course the number incorrect. Multiply the number of correct answers by 4 and subtract the number of incorrect answers to get the total score.

1. A B C **D** E
2. A B C D **E**
3. A **B** C D E
4. A B C **D** E
5. A B C D **E**
6. A B C **D** E
7. A B **C** D E
8. A B C D **E**
9. **A** B C D E
10. A B C D **E**
11. A B **C** D E
12. A B **C** D E
13. A B C **D** E
14. A B **C** D E
15. **A** B C D E
16. A **B** C D E
17. A B C **D** E
18. **A** B C D E
19. A **B** C D E
20. A B **C** D E
21. A B **C** D E
22. A **B** C D E
23. A **B** C D E
24. A B **C** D E
25. A B **C** D E
26. **A** B C D E
27. **A** B C D E
28. A **B** C D E
29. A B C **D** E
30. A **B** C D E